Schrödinger cat states of a non-stationary generalized oscillator*

1996 J. Phys. A: Math. Gen. 29 2091

(http://iopscience.iop.org/0305-4470/29/9/021)

J. Phys. A: Math. Gen. **29** (1996) 2091–2109. Printed in the UK

Schrödinger cat states of a non-stationary generalized oscillator*

O Castaños, R López-Peña and V I Man'ko†
Instituto de Ciencias Nucleares, UNAM, Circuito Exterior, CU, Apdo. Postal 70-543, 04510 Mexico, DF, Mexico

Received 22 August 1995, in final form 4 December 1995

Abstract. The two-mode even and odd coherent Schrödinger cat states are introduced for the generalized stationary and non-stationary harmonic oscillator. The expected values of positions and momenta and their dispersion matrices are calculated for the Schrödinger cat states. The quadrature squeezing and correlation phenomenon are studied and the Wigner and Husimi functions are constructed explicitly for these states. The photon statistics of the two-mode even and odd coherent states of the generalized oscillator is described in terms of the photon distribution function, which is expressed in terms of Hermite polynomials. The photon number, means and dispersions show oscillations which are characteristics of squeezing and correlation phenomenon.

1. Introduction

The even and odd coherent states for one-mode harmonic oscillators were introduced in the 1970s [1]. These states, which have been called Schrödinger cat states [2, 3], were studied in detail [4–6]. These states are representatives of non-classical states [7]. The importance of these states is related to the experimental possibility of their generation in high-quality resonators [8, 9] and in possible applications to improve the sensitivity of the interferometric measurements in the gravitational wave antenna [10] where even and odd coherent light may play an alternative role to squeezed light [11–13]. Some characteristics of the two-mode squeezed states have been studied in [14–17]. Schrödinger cat states have properties similar to those of the squeezed states, i.e. the squeezed vacuum state and the even coherent state contain Fock states with an even number of photons. Until now most attention has been devoted to one-mode Schrödinger cat states. Recently, the polymode generalization of the Schrödinger cat states has been introduced [18] and some aspects of evolution of these states due to time dependence of the parameters of a Hamiltonian, quadratic in positions and momenta, have been discussed [19]. The generalized two-mode oscillator has been studied [20] and some aspects of two-mode squeezing for this generalized oscillator due to its parametric excitation and damping have been considered [21].

The first aim of this paper is to study the properties of the two-mode Schrödinger cat states for the generalized non-stationary oscillator. The physical motivation for this is the necessity to study the behaviour of non-classical states of a two-mode electromagnetic field in a resonator with non-stationary boundaries. As was discussed in [22] (cf [23–25]) there exists a non-stationary Casimir effect which produces the squeezing and correlation

* Work supported in part by project UNAM-DGAPA IN102094.
† On leave from Lebedev Physical Institute, Moscow, Russia.

phenomenon for initially coherent states if the walls of a resonator move and if the media inside the resonator has a time-dependent refractive index. In these conditions, the Casimir forces create the photons from the vacuum state and produce squeezing and correlation of the photon quadrature components. Until now the initial states of the photons in the resonator with moving walls were considered as Gaussian coherent state packets. The second aim of this work is to take the initial state (either even or odd two-mode Schrödinger cat states), and to study its evolution due to the time dependence of the Hamiltonian parameters.

The paper is organized as follows. In section 2 we review the integrals of the motion associated with two-dimensional generalized harmonic oscillator [20, 21]. In section 3 we introduce two-mode Schrödinger cat states for the non-stationary oscillator, which we will dub the generalized Schrödinger cat states (GSCS), and calculate the quadrature components and dispersions in these states. In section 4 we calculate the Husimi and Wigner functions of the even and odd GSCS. In section 5 we obtain the photon distribution function and its evolution for GSCS. For these states we evaluate the mean value of the number of photons and their dispersions. The conditions for squeezing and correlation of GSCS are discussed. Finally, we give a summary of the main properties of even and odd GSCS.

2. Generalized harmonic oscillator

Recently the generalized harmonic oscillator

$$H = \frac{1}{2}\sum_{k=1}^{2}\left(\frac{p_k^2}{m} + m\omega_0^2 q_k^2\right) + \lambda(q_1 p_2 - q_2 p_1) \tag{2.1}$$

was studied. The accidental degeneracy [20] associated with the system was investigated, for m and λ—equal to a rational number—constants, and the linear time-dependent invariants were constructed, including when these parameters are functions of time [21].

The Hamiltonian can be rewritten in terms of creation and annihilation boson operators representing the modes of an electromagnetic field. Thus we have

$$\frac{H}{\hbar\omega_0} = F(t)(N_1 + N_2) + G(t)\big(\boldsymbol{a}^\dagger \cdot \boldsymbol{a}^\dagger + \boldsymbol{a} \cdot \boldsymbol{a}\big) - \mathrm{i}\lambda_0(t)\big(a_1^\dagger a_2 - a_2^\dagger a_1\big) + F(t) \tag{2.2}$$

where

$$N_k = a_k^\dagger a_k \tag{2.3a}$$

$$F(t) = \frac{1}{f}(1 + f^2) \tag{2.3b}$$

$$G(t) = \frac{1}{2f}\left(1 - f^2\right) \tag{2.3c}$$

with the functions $f(t)$ and $\lambda_0(t)$ satisfying the initial conditions $f(0) = \lambda_0(0) = 1$.

To solve the non-stationary Schrödinger equation of the Hamiltonian (2.1) or (2.2), we use the constants of the motion of the system. The resulting invariants can be given in terms of 4-vectors of positions and momenta, or photon creation and annihilation operators:

$$\widehat{\mathbf{Q}}_0(t) = \boldsymbol{\Lambda}\widehat{\mathbf{Q}} \tag{2.4a}$$

$$\widehat{\mathbf{A}}_0(t) = \boldsymbol{M}\widehat{\mathbf{A}} \tag{2.4b}$$

where the following 4-vector operators were defined:

$$\widehat{\mathbf{Q}}_0^{\mathrm{t}}(t) = (p_{10}(t),\ p_{20}(t),\ q_{10}(t),\ q_{20}(t)) \tag{2.5a}$$

$$\widehat{\mathbf{Q}}^{\mathrm{t}} = (p_1,\ p_2,\ q_1,\ q_2) \tag{2.5b}$$

and

$$\widehat{\mathbf{A}}_0^{\mathrm{t}}(t) = (A_1(t),\ A_2(t),\ A_1^{\dagger}(t),\ A_2^{\dagger}(t)) \tag{2.5c}$$

$$\widehat{\mathbf{A}}^{\mathrm{t}} = (a_1,\ a_2,\ a_1^{\dagger},\ a_2^{\dagger})\,. \tag{2.5d}$$

Here and below the superscript t means the transpose of the matrix. The matrices $\mathbf{\Lambda}$ and $\boldsymbol{M}$ are elements of the four-dimensional symplectic group, $Sp(4, \mathbb{R})$, which is acting on the phase space of the generalized harmonic oscillator. Therefore it is constructed, in general, in terms of ten real parameters. It is straightforward to show that the matrix $\boldsymbol{M}$ is related to $\mathbf{\Lambda}$ through

$$\boldsymbol{M} = \mathbf{g}_4\,\mathbf{\Lambda}\ \mathbf{g}_4{}^{-1} \equiv \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \tag{2.6}$$

with $\mathbf{g}_4$ defined by the direct product of matrices $\mathbf{g}_4 = \mathbf{g} \otimes \mathbf{I}_2$, and

$$\mathbf{g} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{i} & 1 \\ -\mathrm{i} & 1 \end{pmatrix}\,. \tag{2.7}$$

For the cases studied in [21], the four-dimensional symplectic matrix $\mathbf{\Lambda}$ is given as the direct product of matrices: $\mathbf{\Lambda} = \boldsymbol{\mu} \otimes\ \mathbf{R}$. The matrix $\boldsymbol{\mu}$ is a two-dimensional symplectic matrix and $\mathbf{R}$ is a rotation matrix, which are denoted by

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix} \qquad \mathbf{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{2.8}$$

where the definition $\theta = \int_0^{\mathrm{t}} \mathrm{d}\tau\ \lambda_0(\tau)$ was used. Then if we know the μ_k's and θ, by means of (2.4) and (2.8) we construct the integrals of motion. It is important to note that the symmetry algebra associated with this decomposition of $\mathbf{\Lambda}$, a symplectic matrix in four dimensions, corresponds to the direct product of matrices which depends only on four parameters, because the determinant of $\boldsymbol{\mu}$ is equal to unity. For this model of the generalized harmonic oscillator, the $\mathbf{\Lambda}$ matrix forms part of the subgroup $Sp(2,\ \mathbb{R}) \otimes SO(2)$ of the general linear symplectic group $Sp(4,\ \mathbb{R})$.

Next we write the analytic expressions of the μ_k's for the case where $\lambda_0(t)$ is an arbitrary function of time and

$$f(t) = \exp(\gamma t)\,. \tag{2.9}$$

In principle, these choices of parameters can describe phenomenologically the behaviour of the electromagnetic field inside a cavity with different regimes of moving walls and a time-varying dielectric medium. Thus we get

$$\mu_1 = \exp\{-\gamma t/2\}\left(\cos\Omega t + \frac{\gamma}{2}\frac{\sin\Omega t}{\Omega}\right) \tag{2.10a}$$

$$\mu_2 = \exp\{\gamma t/2\}\frac{\sin\Omega t}{\Omega} \tag{2.10b}$$

$$\mu_3 = -\exp\{-\gamma t/2\}\ \frac{\sin\Omega t}{\Omega} \tag{2.10c}$$

$$\mu_4 = \exp\{\gamma t/2\}\left(\cos\Omega t - \frac{\gamma}{2}\frac{\sin\Omega t}{\Omega}\right) \tag{2.10d}$$

with $\Omega^2 = 1 - \gamma^2/4$. It is important to emphasize that (2.10) can also be used when $\gamma^2 \geqslant 4$ by making the appropriate replacements and limit procedures [21].

3. Generalized cat states and quadrature components

The expressions of the correlated states for the generalized two-dimensional harmonic oscillator are given by the normalized wavefunction [21, 29]

$$\Phi_\alpha(\boldsymbol{q},t) = \exp\left\{-\frac{|\boldsymbol{\alpha}|^2}{2} + \frac{1}{2}\frac{\mu_3 - \mathrm{i}\mu_1}{\mu_3 + \mathrm{i}\mu_1}\boldsymbol{\alpha}\cdot\boldsymbol{\alpha} + \frac{\mathrm{i}\sqrt{2}}{\mu_3 + \mathrm{i}\mu_1}\boldsymbol{q}\,\widetilde{\mathbf{R}}\boldsymbol{\alpha}\right\}\Phi_0(\boldsymbol{q},t) \tag{3.1}$$

where the ground-like wavefunction is

$$\Phi_0(\boldsymbol{q},t) = \frac{1}{\sqrt{\pi}(\mu_3 + \mathrm{i}\mu_1)}\exp\left\{-\frac{\mathrm{i}}{2}\frac{\mu_4 + \mathrm{i}\mu_2}{\mu_3 + \mathrm{i}\mu_1}\boldsymbol{q}\cdot\boldsymbol{q}\right\}. \tag{3.2}$$

This wavefunction is the expression for the generalized correlated state in the coordinate representation of the discussed physical system. By making the appropriate substitutions of the μ_k and θ, we get the corresponding solutions for the generalized harmonic oscillator.

The even and odd GSCS are constructed by the linear combinations [1, 18, 19]

$$\Phi_{\alpha_\pm}(\boldsymbol{q},\ t) = \mathcal{N}_\pm[\Phi_\alpha(\boldsymbol{q},\ t) \pm \Phi_{-\alpha}(\boldsymbol{q},\ t)] \tag{3.3a}$$

where $\mathcal{N}_\pm$ are the normalization constants:

$$\mathcal{N}_+ = \frac{\exp[(|\alpha_1|^2 + |\alpha_2|^2)/2]}{2\sqrt{\cosh(|\alpha_1|^2 + |\alpha_2|^2)}} \tag{3.3b}$$

$$\mathcal{N}_- = \frac{\exp[(|\alpha_1|^2 + |\alpha_2|^2)/2]}{2\sqrt{\sinh(|\alpha_1|^2 + |\alpha_2|^2)}}. \tag{3.3c}$$

It is important to emphasize that these states satisfy the time-dependent Schrödinger equation of the Hamiltonian (2.1).

Let us calculate the quadrature components for these states. The dispersion matrix is given by

$$\sigma_{\alpha\beta} = \tfrac{1}{2}\langle\{Q_\alpha,\ Q_\beta\}\rangle - \langle Q_\alpha\rangle\langle Q_\beta\rangle$$

where $\{Q_\alpha,\ Q_\beta\}$ denotes the anticommutator between operators Q_α and Q_β. Using expression (2.4a) for the linear invariants, the last expression can be rewritten for even and odd GSCS as

$$\boldsymbol{\sigma}^\pm(t) = \boldsymbol{\Lambda}^{-1}(t)\,\boldsymbol{\sigma}^\pm(0)\left(\boldsymbol{\Lambda}^{-1}(t)\right)^{\mathrm{t}}$$

where $\boldsymbol{\sigma}^\pm(0)$ is the dispersion matrix for the initial conditions, i.e. the quadrature components for the standard even and odd Schrödinger cat states. Because $\boldsymbol{\Lambda}$ is a symplectic matrix, its inverse is given by $\boldsymbol{\Lambda}^{-1} = -\boldsymbol{\Sigma}\boldsymbol{\Lambda}^{\mathrm{t}}\boldsymbol{\Sigma}$, where $\boldsymbol{\Sigma}$ is the symplectic metric. Substituting this result into the previous expression, we obtain

$$\begin{aligned}\boldsymbol{\sigma}^\pm(t) &= \boldsymbol{\Sigma}\boldsymbol{\Lambda}^{\mathrm{t}}\boldsymbol{\Sigma}\,\boldsymbol{\sigma}^\pm(0)\,\boldsymbol{\Sigma}\boldsymbol{\Lambda}\boldsymbol{\Sigma}\\ &= \left(\boldsymbol{\mu}^{-1}\otimes\mathbf{R}^{\mathrm{t}}\right)\,\boldsymbol{\sigma}^\pm(0)\,\left(\left(\boldsymbol{\mu}^{-1}\right)^{\mathrm{t}}\otimes\mathbf{R}\right).\end{aligned} \tag{3.4}$$

In the last step the expression for matrix $\mathbf{\Lambda}$ in terms of matrices $\boldsymbol{\mu}$ and $\mathbf{R}$ given in (2.8) was used. We can evaluate expression (3.4), and get analytic expressions for the dispersion in positions and momenta:

$$\boldsymbol{\sigma}_{pp}^{\pm}(t) = \widetilde{\boldsymbol{R}}\left\{\mu_4^2\boldsymbol{\sigma}_{pp}^{\pm}(0) - \mu_2\mu_4\boldsymbol{\sigma}_{pq}^{\pm}(0) - \mu_2\mu_4\boldsymbol{\sigma}_{qp}^{\pm}(0) + \mu_2^2\boldsymbol{\sigma}_{qq}^{\pm}(0)\right\}\boldsymbol{R} \tag{3.5a}$$

$$\boldsymbol{\sigma}_{pq}^{\pm}(t) = \widetilde{\boldsymbol{R}}\left\{-\mu_3\mu_4\boldsymbol{\sigma}_{pp}^{\pm}(0) + \mu_1\mu_4\boldsymbol{\sigma}_{pq}^{\pm}(0) + \mu_2\mu_3\boldsymbol{\sigma}_{qp}^{\pm}(0) - \mu_1\mu_2\boldsymbol{\sigma}_{qq}^{\pm}(0)\right\}\boldsymbol{R} \tag{3.5b}$$

$$\boldsymbol{\sigma}_{qq}^{\pm}(t) = \widetilde{\boldsymbol{R}}\left\{\mu_3^2\boldsymbol{\sigma}_{pp}^{\pm}(0) - \mu_1\mu_3\boldsymbol{\sigma}_{pq}^{\pm}(0) - \mu_1\mu_3\boldsymbol{\sigma}_{qp}^{\pm}(0) + \mu_1^2\boldsymbol{\sigma}_{qq}^{\pm}(0)\right\}\boldsymbol{R} \tag{3.5c}$$

where the $\boldsymbol{\sigma}^{\pm}(0)$ denote the dispersion matrices of two-mode Schrödinger cat states. These can be easily calculated:

$$\left(\boldsymbol{\sigma}_{\mathrm{pp}}^{\pm}\right)_{ij}(0) = \mathrm{Re}\left(\alpha_i\left(-\alpha_j + \Delta_{\pm}\alpha_j^*\right)\right) + \tfrac{1}{2}\delta_{ij} \tag{3.6a}$$

$$\left(\boldsymbol{\sigma}_{\mathrm{pq}}^{\pm}\right)_{ij}(0) = \mathrm{Im}\left(\alpha_i\left(\alpha_j + \Delta_{\pm}\alpha_j^*\right)\right) = \left(\boldsymbol{\sigma}_{\mathrm{qp}}^{\pm}\right)_{ji}(0) \tag{3.6b}$$

$$\left(\boldsymbol{\sigma}_{\mathrm{qq}}^{\pm}\right)_{ij}(0) = \mathrm{Re}\left(\alpha_i\left(\alpha_j + \Delta_{\pm}\alpha_j^*\right)\right) + \tfrac{1}{2}\delta_{ij}\,. \tag{3.6c}$$

In the last expressions $\Delta_{\pm}$ is defined as

$$\Delta_{\pm} = \begin{cases} \tanh|\boldsymbol{\alpha}|^2 & \text{for even cats} \\ \coth|\boldsymbol{\alpha}|^2 & \text{for odd cats}\,. \end{cases} \tag{3.7}$$

The dispersion matrices (3.5) were calculated for the parameters $\lambda = 1$ and $\gamma = 0.1$. The quadrature dispersions for the amplitudes $\alpha_1 = 0.2 + 0.1\mathrm{i}$ and $\alpha_2 = 0.3 + 0.1\mathrm{i}$ of the even and odd GSCS are displayed in figures 1–6. The behaviour of the even state is illustrated in figures 1–3, while the odd case is shown in figures 4 and 5.

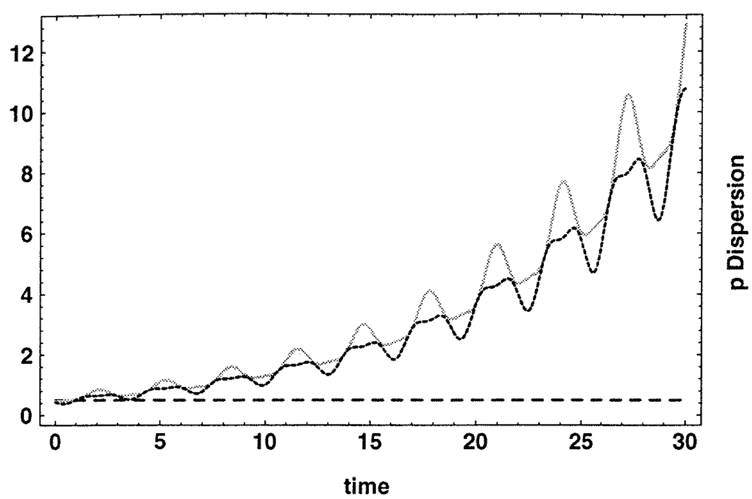

Figure 1. Quadrature dispersions for the amplitudes $\alpha_1 = 0.2 + \mathrm{i}0.1$ and $\alpha_2 = 0.3 + \mathrm{i}0.1$ of the even state. The dispersions along the first and second directions in the momenta are plotted, with full grey and short-broken curves, respectively. The value $\frac{1}{2}$ is displayed with a long-broken line as reference.

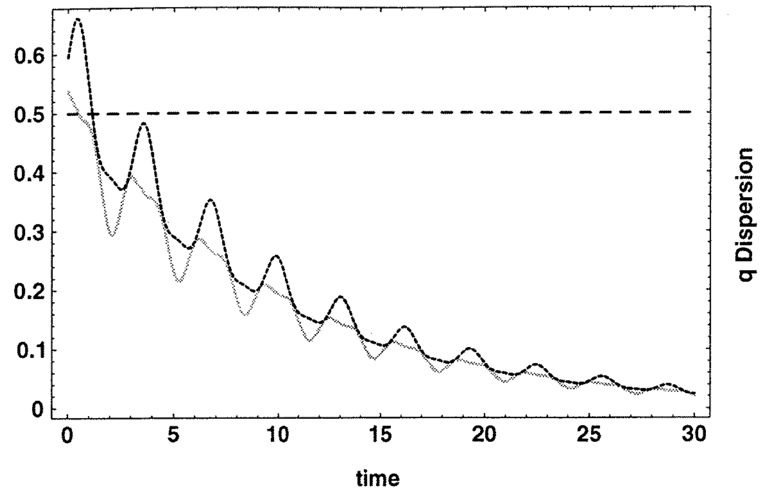

Figure 2. Quadrature dispersions for the even state, using the same values indicated in figure 1. The dispersions along the first and second directions in the positions are plotted, again with full grey and short-broken curves, respectively. The value $\frac{1}{2}$ is displayed with a long-broken line as before.

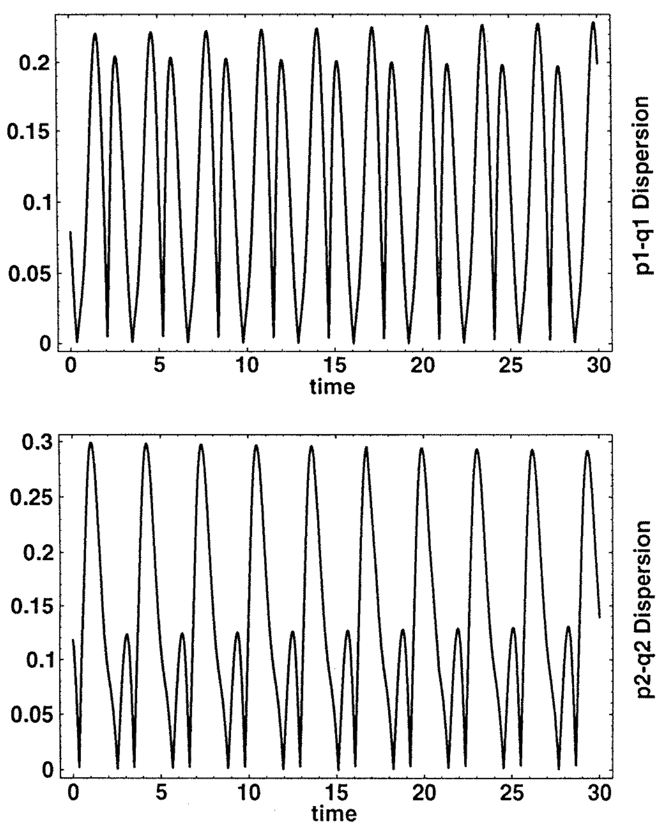

Figure 3. Correlation coefficients between momenta and positions in the same direction for the even state, with the same values as in figure 1.

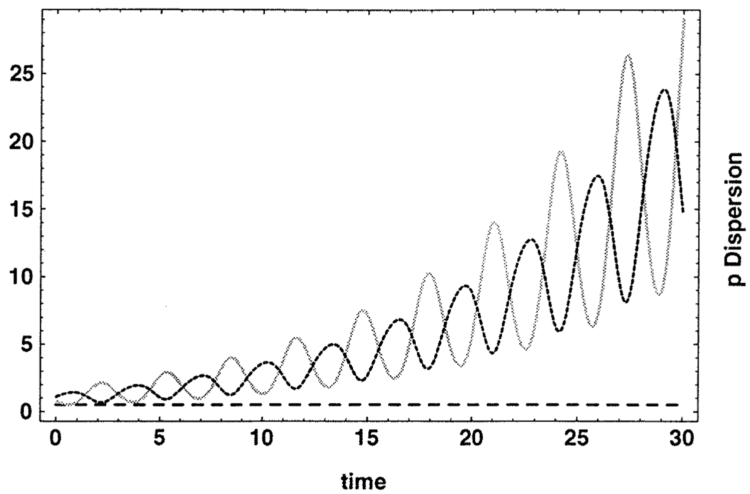

Figure 4. Quadrature dispersions for the odd state. The dispersions along the first (full grey curve) and second (short-broken curve) directions in the momenta are plotted. The parameters of the model and amplitudes of the state are as in figure 1. The value $\frac{1}{2}$ is displayed with a long-broken line.

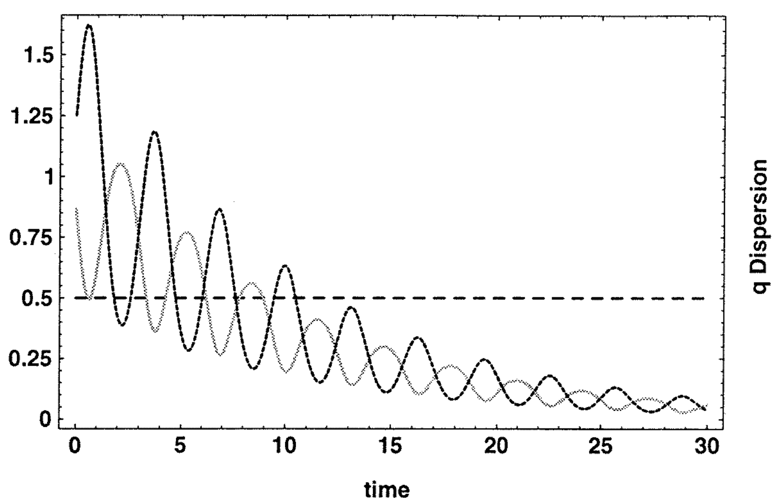

Figure 5. Quadrature dispersions for the odd state, using the same values as in figure 1. The dispersions along the first (full grey curve) and second (short-broken curve) directions in the momenta are plotted. The value $\frac{1}{2}$ is displayed with a long-broken line.

The dispersions of the momenta in directions one and two are displayed in figure 1. These are increasing oscillatory functions of time. Only for a very small interval of time is there squeezing for the momentum in direction one. The correlation coefficient

$$r = \left|\sigma_{p_1 p_2}\right| / \sqrt{\sigma_{p_1 p_1}\sigma_{p_2 p_2}}$$

has a periodic behaviour in time. It performs an oscillatory motion with respect to time, and ranges between 0 and 0.1.

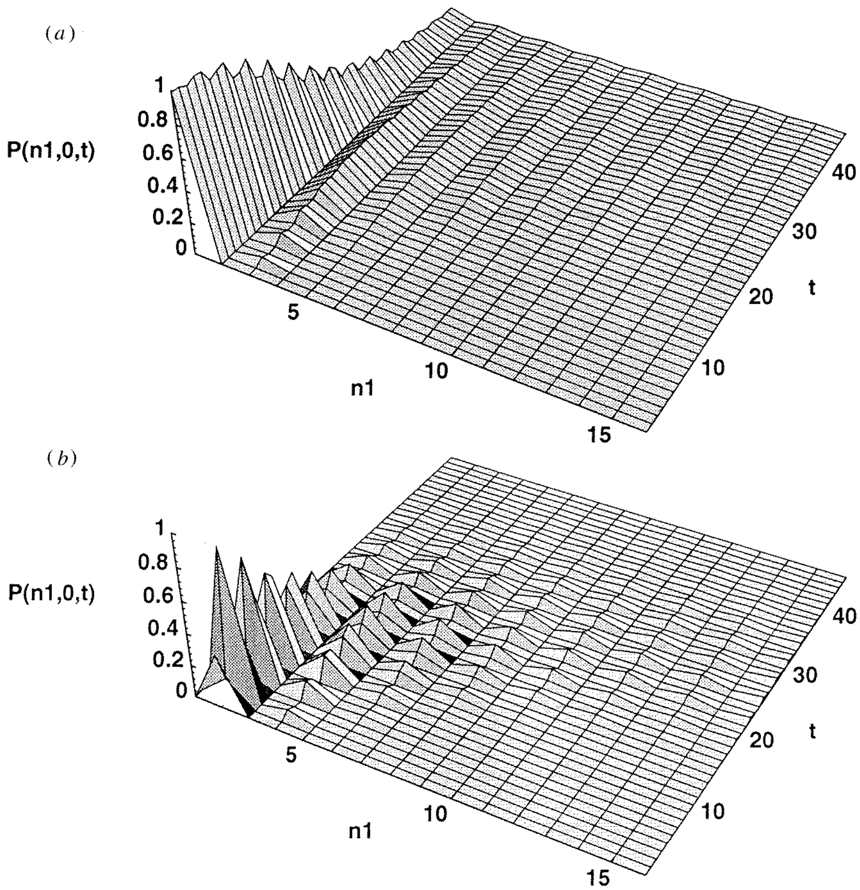

Figure 6. Evolution of the first mode in the photon distribution function of the generalized Schrödinger cat states, for the amplitudes $\alpha_1 = 0.2 + \mathrm{i}0.1$ and $\alpha_2 = 0.3 + \mathrm{i}0.1$. The even is displayed in (*a*), and the odd in (*b*). The parameters of the model are $\lambda = 1$ and $\gamma = 0.1$.

In figure 2, the position dispersions are shown. These are decreasing functions of time and the phenomenon of squeezing is clearly established in both directions. The corresponding correlation coefficient for the positions has a periodic behaviour, similar to that of the momenta. Both show a low correlation of the components.

It is worthwhile noting that the quadrature dispersions in both directions exhibit roughly the same position of local minima and maxima.

The correlation coefficients between the positions and momenta are displayed in figure 3. For q_1, p_1 and q_2, p_2, the correlations are periodic functions of time, but with different ranges, varying from 0 to 0.22, and from 0 to 0.3, respectively. For the q_1, p_2 and q_2, p_1 cases, the correlation coefficients are oscillatory functions with similar ranges, varying from 0.01 to 0.14.

The dispersions of momenta for odd GSCS in directions one and two are displayed in figure 4. These are faster increasing oscillatory functions of time than for the even case, without showing the phenomenon of squeezing. The correlation coefficient between the momentum components performs an oscillatory motion, and for most of the time presents a strong correlation.

In figure 5, the corresponding dispersions for the positions are shown. These are decreasing functions of time and the phenomenon of squeezing occurs in an oscillatory manner for $t \lesssim 9$. After that interval the squeezing is permanent in both directions. The corresponding correlation coefficient for the positions also displays a strong correlation, similar to that of the momenta. The correlation coefficients between the positions and momenta present an oscillatory behaviour with similar ranges, from 0 to 0.18. For q_1, p_1, q_2, p_2 and q_2, p_1, the correlation coefficients found have the same order of magnitude, while for q_1, p_2, the coefficient has approximately one third of the magnitude of the others.

In summary, figures 1–5 show the difference in the behaviour of the even and odd GSCS. They are squeezed and correlated states which can be considered as an alternative in the development and research on gravitational waves or in the study of electromagnetic fields inside cavities with moving walls.

4. Husimi and Wigner functions

There are different ways to generate a probability density for a physical system. In this paper, we will study the Husimi and Wigner functions for the Schrödinger cat states of the generalized oscillator.

The Husimi function is the matrix element of the density operator $\rho_\pm = |\boldsymbol{\alpha}_\pm\rangle\langle\boldsymbol{\alpha}_\pm|$ in the coherent state basis $|\boldsymbol{z}\rangle$, with $\boldsymbol{z} = (\boldsymbol{q}+\mathrm{i}\boldsymbol{p})/\sqrt{2}$. For the generalized harmonic oscillator, the Husimi function is

$$Q_\alpha(\boldsymbol{q}, \boldsymbol{p}, t) = \langle \boldsymbol{z}|\widehat{\rho}|\boldsymbol{z}\rangle = \langle \boldsymbol{z}|\boldsymbol{\alpha}, t\rangle\langle\boldsymbol{\alpha}, t|\boldsymbol{z}\rangle = \left|G(\boldsymbol{z}^*, \boldsymbol{\alpha}, t)\right|^2 \tag{4.1}$$

where $G(\boldsymbol{z}^*, \boldsymbol{\alpha}, t) \equiv \langle \boldsymbol{z}|U(t)|\boldsymbol{\alpha}\rangle$ is the corresponding Green function. For any homogeneous quadratic Hamiltonian, the propagator in the coherent state representation takes the form [29]

$$G(\boldsymbol{z}^*, \boldsymbol{\gamma}, t) = \frac{\exp(-|\boldsymbol{z}|^2/2 - |\boldsymbol{\gamma}|^2/2)}{\sqrt{\det M_1}} \exp\left(-\tfrac{1}{2}\boldsymbol{z}^* M_1^{-1} M_2 \boldsymbol{z}^* + \boldsymbol{z}^* M_1^{-1}\boldsymbol{\gamma} + \tfrac{1}{2}\boldsymbol{\gamma} M_3 M_1^{-1}\boldsymbol{\gamma}\right) \tag{4.2}$$

where the matrices M_k are determined by expression (2.6).

The GSCS yield a Husimi function defined by the expression

$$\begin{aligned} Q_{\alpha\pm}(\boldsymbol{q}, \boldsymbol{p}, t) &= \langle \boldsymbol{z}|\widehat{\rho}_\pm|\boldsymbol{z}\rangle \\ &= |\mathcal{N}_\pm|^2 \left\{ \left|G(\boldsymbol{z}^*, \boldsymbol{\alpha}, t)\right|^2 \pm 2\,\mathrm{Re}\left\{G(\boldsymbol{z}^*, -\boldsymbol{\alpha}, t) G^*(\boldsymbol{z}^*, \boldsymbol{\alpha}, t)\right\} \right. \\ &\quad \left. + \left|G(\boldsymbol{z}^*, -\boldsymbol{\alpha}, t)\right|^2 \right\}. \end{aligned} \tag{4.3}$$

In the generalized harmonic oscillator case, the M_k's matrices take the form [29]

$$\sqrt{\det M_1} \equiv g_1(\mu_k) = \tfrac{1}{2}(\mu_1 + \mu_4 - \mathrm{i}\mu_3 + \mathrm{i}\mu_2) \tag{4.4a}$$

$$M_1^{-1} M_2 \equiv g_2(\mu_k) I_2 = \frac{\{(\mu_4 + \mathrm{i}\mu_3)^2 + (\mathrm{i}\mu_1 + \mu_2)^2\}}{g(\mu_k)} I_2 \tag{4.4b}$$

$$M_1^{-1} \equiv g_3(\mu_k)\widetilde{\mathbf{R}} = 2\frac{(\mu_1 + \mu_4 + \mathrm{i}\mu_3 - \mathrm{i}\mu_2)}{g(\mu_k)}\widetilde{\mathbf{R}} \tag{4.4c}$$

$$M_3 M_1^{-1} \equiv g_4(\mu_k) I_2 = \frac{\{(\mu_4 - \mathrm{i}\mu_2)^2 + (-\mathrm{i}\mu_1 + \mu_3)^2\}}{g(\mu_k)} I_2 \tag{4.4d}$$

with

$$g(\mu_k) = (\mu_1 + \mu_4)^2 + (\mu_3 - \mu_2)^2 . \tag{4.4e}$$

Substituting these relations into (4.2), we obtain the propagator for the generalized harmonic oscillator:

$$G(\boldsymbol{z}^*, \boldsymbol{\gamma}, t) = \frac{\exp(-|\boldsymbol{z}|^2/2 - |\boldsymbol{\gamma}|^2/2)}{g_1} \exp\left(-\tfrac{1}{2} g_2 \boldsymbol{z}^* \cdot \boldsymbol{z}^* + g_3 \boldsymbol{z}^* \widetilde{\mathbf{R}} \boldsymbol{\gamma} + \tfrac{1}{2} g_4 \boldsymbol{\gamma} \cdot \boldsymbol{\gamma}\right) . \tag{4.5}$$

Introducing (4.5) into (4.3), we get the Husimi function for the even and odd GSCS:

$$\begin{aligned} Q_{\alpha\pm}(\boldsymbol{q}, \boldsymbol{p}, t) = {} & |\mathcal{N}_\pm|^2 \frac{2}{|g_1|^2} \exp\left\{-|\boldsymbol{z}|^2 - |\boldsymbol{\alpha}|^2 + \mathrm{Re}\left(g_4 \boldsymbol{\alpha} \cdot \boldsymbol{\alpha} - g_2^* \boldsymbol{z} \cdot \boldsymbol{z}\right)\right\} \\ & \times \{\cosh[2\,\mathrm{Re}(g_3 \boldsymbol{z}^* \widetilde{\mathbf{R}} \boldsymbol{\alpha})] \pm \cos[2\,\mathrm{Im}\left(g_3 \boldsymbol{z}^* \widetilde{\mathbf{R}} \boldsymbol{\alpha}\right)]\} . \end{aligned} \tag{4.6}$$

The Wigner function is obtained through the evaluation of the matrix element of the density operator associated with the GSCS in the coordinate representation, i.e.

$$\langle \boldsymbol{q} | \widehat{\rho}_\pm | \boldsymbol{q}' \rangle = \langle \boldsymbol{q} | \boldsymbol{\alpha}_\pm, t \rangle \, \langle \boldsymbol{\alpha}_\pm, t | \boldsymbol{q}' \rangle \tag{4.7}$$

where the right-hand side is given by the generalized correlated states in the coordinate representation, equation (3.3). Then from (4.7) we get the Wigner function by making the integrals

$$W_{\alpha\pm}(\boldsymbol{q}, \boldsymbol{p}, t) = \int_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{u} \exp(-\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{u}) \; \Phi_{\alpha\pm}(\boldsymbol{q} + \boldsymbol{u}/2, t) \; \Phi^*_{\alpha\pm}(\boldsymbol{q} - \boldsymbol{u}/2, t) . \tag{4.8}$$

Substituting (3.3) into the last expression, we get a Gaussian integral in the variable $\boldsymbol{u}$, which can be calculated. Instead of this, we use the following procedure. We introduce the function

$$W_{\alpha\beta}(\boldsymbol{q}, \boldsymbol{p}, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{u} \exp(-\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{u}) \, \Phi_\alpha (\boldsymbol{q} + \boldsymbol{u}/2, t) \, \Phi^*_\beta (\boldsymbol{q} - \boldsymbol{u}/2, t) \tag{4.9}$$

in terms of the Weyl–Wigner transform of the density matrix associated with the wavefunction of the generalized harmonic oscillator (3.1). The action of the integrals of motion $\boldsymbol{A}(t)$ and $\boldsymbol{A}^\dagger(t)$, given in (2.4*b*), on the density matrix operator of the generalized oscillator, $\hat{\rho}_{\alpha\beta} = |\boldsymbol{\alpha}, t\rangle \, \langle \boldsymbol{\beta}, t|$, is

$$\boldsymbol{A}(t) \, \hat{\rho}_{\alpha\beta} = \boldsymbol{\alpha} \, \hat{\rho}_{\alpha\beta} \tag{4.10a}$$

$$\hat{\rho}_{\alpha\beta} \, \boldsymbol{A}^\dagger(t) = \boldsymbol{\beta}^* \, \hat{\rho}_{\alpha\beta} . \tag{4.10b}$$

These expressions are written in the Weyl–Wigner representation as

$$\left\{\boldsymbol{z}_0(t) + \frac{1}{2} \frac{\partial}{\partial \boldsymbol{z}_0^*(t)}\right\} W_{\alpha\beta} = \boldsymbol{\alpha} \, W_{\alpha\beta} \tag{4.11a}$$

$$\left\{\boldsymbol{z}_0^*(t) + \frac{1}{2} \frac{\partial}{\partial \boldsymbol{z}_0(t)}\right\} W_{\alpha\beta} = \boldsymbol{\beta}^* \, W_{\alpha\beta} \tag{4.11b}$$

where we have defined the 4-vector

$$Z_0^{\mathrm{t}}(t) = (\boldsymbol{z}_0^{t}(t),\ \boldsymbol{z}_0^{*t}(t)) \tag{4.12}$$

through the relation

$$Z_0(t) = \boldsymbol{g}_4\boldsymbol{\Lambda}\begin{pmatrix}\boldsymbol{p}\\ \boldsymbol{q}\end{pmatrix}. \tag{4.13}$$

The solution of the set of coupled partial differential equations (4.11), together with the normalization condition

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{p}\ W_{\alpha\beta}(\boldsymbol{q},\ \boldsymbol{p},\ t) = \langle\boldsymbol{\beta},\ t|\boldsymbol{\alpha},\ t\rangle$$

gives the result for the function (4.9):

$$\begin{aligned} W_{\alpha\beta}(\boldsymbol{q},\ \boldsymbol{p},\ t) = 2^2 \exp\big(&-2\boldsymbol{z}_0(t)\cdot\boldsymbol{z}_0^*(t) + 2\boldsymbol{\alpha}\cdot\boldsymbol{z}_0^*(t) + 2\boldsymbol{\beta}^*\cdot\boldsymbol{z}_0(t)\big)\\ &\times \exp\big(-\boldsymbol{\alpha}\cdot\boldsymbol{\beta}^* - |\boldsymbol{\alpha}|^2/2 - |\boldsymbol{\beta}|^2/2\big). \end{aligned} \tag{4.14}$$

Then the Wigner function for the Schrödinger cat states for any homogeneous quadratic Hamiltonian system is a linear combination of expressions (4.14):

$$\begin{aligned} W_{\pm}(\boldsymbol{q},\ \boldsymbol{p},\ t) = |\mathcal{N}_{\pm}|^2\big\{&W_{\alpha,\alpha}(\boldsymbol{q},\ \boldsymbol{p},\ t) \pm\ W_{\alpha,-\alpha}(\boldsymbol{q},\ \boldsymbol{p},\ t)\\ &\pm W_{-\alpha,\alpha}(\boldsymbol{q},\ \boldsymbol{p},\ t) + W_{-\alpha,-\alpha}(\boldsymbol{q},\ \boldsymbol{p},\ t)\big\}. \end{aligned} \tag{4.15}$$

For the generalized harmonic oscillator the expression (4.14) is given by

$$\begin{aligned} W_{\alpha\beta}(\boldsymbol{q},\ \boldsymbol{p},\ t) = 4\exp\Bigg\{&-|\boldsymbol{\alpha}|^2/2 - |\boldsymbol{\beta}|^2/2 - \boldsymbol{\alpha}\cdot\boldsymbol{\beta}^* - (\boldsymbol{p}^{\,t},\ \boldsymbol{q}^{\,t})\left\{\boldsymbol{\mu}^{\mathrm{t}}\mathbf{g}^{\mathrm{t}}\boldsymbol{\sigma}_x\,\mathbf{g}\,\boldsymbol{\mu}\otimes\mathbf{I}_2\right\}\begin{pmatrix}\boldsymbol{p}\\ \boldsymbol{q}\end{pmatrix}\\ &+2\left(\boldsymbol{\alpha}^{t},\ \boldsymbol{\beta}^{*t}\right)\{\boldsymbol{\sigma}_x\mathbf{g}\boldsymbol{\mu}\otimes\mathbf{R}\}\begin{pmatrix}\boldsymbol{p}\\ \boldsymbol{q}\end{pmatrix}\Bigg\} \end{aligned} \tag{4.16}$$

where $\boldsymbol{\sigma}_x$ is the first component of the standard Pauli matrices. The amplitude $\boldsymbol{\alpha}$ is determined by the initial values of the quadrature components through the relation

$$\langle\widehat{\mathbf{A}}_0^{\mathrm{t}}(t)\rangle = \left(\boldsymbol{\alpha}^{t},\ \boldsymbol{\alpha}^{*t}\right).$$

5. Collapses and revivals in Schrödinger cat states

In this section we obtain the photon distribution for the generalized Schrödinger cat states. This is done by calculating the transition amplitudes from GSCS to number states

$$\langle n_1,\ n_2|\gamma_1\,,\gamma_2,\ t\rangle_{\pm} = \mathcal{N}_{\pm}\big(\langle n_1,\ n_2|\gamma_1,\ \gamma_2,\ t\rangle \pm \langle n_1,\ n_2| - \gamma_1,\ -\gamma_2,\ t\rangle\big). \tag{5.1}$$

To calculate the right-hand side of the last equation, it is convenient to expand the propagator given in the coherent state representation, $G(\boldsymbol{z}^*,\ \boldsymbol{\gamma},\ t) \equiv \langle\boldsymbol{z}\,|\mathrm{U(t)}|\boldsymbol{\gamma}\rangle$, in terms of generalized correlated states in the Fock space,

$$\langle\boldsymbol{z}\,|U(t)|\boldsymbol{\gamma}\rangle = \sum_{m_1,\,m_2}\langle z_1,\ z_2|m_1,\ m_2\rangle\ \langle m_1,\ m_2|\mathrm{U(t)}|\gamma_1,\ \gamma_2\rangle\,. \tag{5.2}$$

We compare the power-series expansion of the propagator with the last expression to get the probability amplitude for n_1 and n_2 photons in the generalized correlated state $|\boldsymbol{\gamma}, t\rangle$ [29]. For the generalized harmonic oscillator, we get

$$\langle n_1, n_2|\boldsymbol{\gamma}, t\rangle = \frac{1}{g_1\sqrt{n_1!n_2!}} \exp\left(-\tfrac{1}{2}|\boldsymbol{\gamma}|^2 + \tfrac{1}{2}g_4\boldsymbol{\gamma}\cdot\boldsymbol{\gamma}\right)\left(\tfrac{1}{2}g_2\right)^{(n_1+n_2)/2} \times H_{n_1}\left(\frac{g_3}{\sqrt{2g_2}}\left(\gamma_1\cos\theta - \gamma_2\sin\theta\right)\right)H_{n_2}\left(\frac{g_3}{\sqrt{2g_2}}\left(\gamma_1\sin\theta + \gamma_2\cos\theta\right)\right). \tag{5.3}$$

The photon-number distribution $P^{\pm}_{n_1,n_2}(\boldsymbol{\gamma}, t) = |\langle n_1, n_2|\boldsymbol{\gamma}, t\rangle_{\pm}|^2$ is

$$P^{\pm}_{n_1,n_2}(\gamma_1, \gamma_2, t) = |\mathcal{N}_{\pm}|^2\, 2\left\{1 \pm (-1)^{n_1+n_2}\right\} |\langle n_1, n_2|\gamma_1, \gamma_2, t\rangle|^2 . \tag{5.4}$$

According to this expression, the photon-number distribution function preserves evenness of the number of photons for the even state, and oddness, for the odd state.

The photon distribution functions were calculated for even and odd GSCS with amplitudes $\alpha_1 = 0.2 + 0.1\mathrm{i}$ and $\alpha_2 = 0.3 + 0.1\mathrm{i}$. Figures 6(*a*) and (*b*) show how many photons in the first mode are contained in the mentioned even and odd cat states as a function of time, respectively. These distributions have an oscillatory motion, in which the phenomenon of collapses and revivals are present. The behaviour of the photon distribution function with respect to the number of photons in the second mode for the even and odd GSCS is very similar to those illustrated in figure 6. To have a different result, it is necessary to select very different amplitudes α_1 and α_2 for the generalized Schrödinger cat states.

For any homogeneous quadratic Hamiltonian, the expectation values of the number of photons in the first and second modes, $\langle N_1\rangle_{\pm}$ and $\langle N_2\rangle_{\pm}$, respectively, together with their mean squared fluctuations, can be evaluated using the expression of the creation and annihilation photon operators in terms of the integrals of motion, and their commutation properties. However, it is simpler to use the characteristic function method (cf the appendix). The mean values of the number of photons are

$$\langle N_j\rangle_{\pm} = \tfrac{1}{2}\left(\left(\boldsymbol{\sigma}^{\pm}_{pp}(t)\right)_{jj} + \left(\boldsymbol{\sigma}^{\pm}_{qq}(t)\right)_{jj} - 1\right). \tag{5.5}$$

The mean-squared fluctuations can be expressed in terms of the dispersion matrices and expected values of position and momentum generalized correlated eigenstates of the system, the normalization factors defined in (3.3), the exponential

$$\mathcal{C}_{\Gamma}(0) = \exp\left(-2|\alpha_1|^2 - 2|\alpha_2|^2\right)$$

and a 4-vector $S_{\alpha}(\Gamma)$:

$$\begin{aligned}\left(\Delta N_j\right)^2_{\pm} = \tfrac{1}{2}&\left(\left(\left(\boldsymbol{\sigma}_{pp}(t)\right)_{jj} + \left(\boldsymbol{\sigma}_{qq}(t)\right)_{jj}\right)^2 - 1\right) + |\mathcal{N}_{\pm}|^2\left\{2\left(\boldsymbol{\sigma}_{pp}(t)\right)_{jj}\langle p_j\rangle^2\right.\\ &+ 2\left(\boldsymbol{\sigma}_{qq}(t)\right)_{jj}\langle q_j\rangle^2 + 4\left(\boldsymbol{\sigma}_{qp}(t)\right)_{jj}\langle q_j\rangle\langle p_j\rangle + \tfrac{1}{2}\left(\langle p_j\rangle^2 + \langle q_j\rangle^2\right)^2\Big\}\\ &\pm |\mathcal{N}_{\pm}|^2\,\mathcal{C}_{\Gamma}(0)\left\{2\left(\boldsymbol{\sigma}_{pp}(t)\right)_{jj}S_j^2(\Gamma) + 2\left(\boldsymbol{\sigma}_{qq}(t)\right)_{jj}S^2_{2+j}(\Gamma)\right.\\ &+ 4\left(\boldsymbol{\sigma}_{qp}(t)\right)_{jj}S_{2+j}(\Gamma)S_j(\Gamma) + \tfrac{1}{2}\left(S^2_{2+j}(\Gamma) + S^2_j(\Gamma)\right)^2\Big\}\\ &- |\mathcal{N}_{\pm}|^4\left\{\langle p_j\rangle^2 + \langle q_j\rangle^2 \pm \mathcal{C}_{\Gamma}(0)\left(S^2_{2+j}(\Gamma) + S^2_j(\Gamma)\right)\right\}^2 .\end{aligned} \tag{5.6}$$

For the generalized harmonic oscillator, the dispersion matrices are given by [29]

$$\big(\boldsymbol{\sigma}_{pp}(t)\big)_{ij} = \tfrac{1}{2}\big(\mu_2^2 + \mu_4^2\big)\delta_{ij} \tag{5.7a}$$

$$\big(\boldsymbol{\sigma}_{qp}(t)\big)_{ij} = -\tfrac{1}{2}\big(\mu_1\mu_2 + \mu_3\mu_4\big)\delta_{ij} \tag{5.7b}$$

$$\big(\boldsymbol{\sigma}_{qq}(t)\big)_{ij} = \tfrac{1}{2}\big(\mu_1^2 + \mu_3^2\big)\delta_{ij}\,. \tag{5.7c}$$

The mean values of the positions and momenta take the values [29]

$$\begin{aligned}
\langle p_1\rangle &= -\mathrm{i}\frac{1}{\sqrt{2}}\,(\mu_3 - \mathrm{i}\mu_1)\,(\alpha_1\cos\theta - \alpha_2\sin\theta) + \mathrm{CC}\\
\langle p_2\rangle &= -\mathrm{i}\frac{1}{\sqrt{2}}\,(\mu_3 - \mathrm{i}\mu_1)\,(\alpha_1\sin\theta + \alpha_2\cos\theta) + \mathrm{CC}\\
\langle q_1\rangle &= \mathrm{i}\frac{1}{\sqrt{2}}\,(\mu_4 - \mathrm{i}\mu_2)\,(\alpha_1\cos\theta - \alpha_2\sin\theta) + \mathrm{CC}\\
\langle q_2\rangle &= \mathrm{i}\frac{1}{\sqrt{2}}\,(\mu_4 - \mathrm{i}\mu_2)\,(\alpha_1\sin\theta + \alpha_2\cos\theta) + \mathrm{CC}
\end{aligned} \tag{5.8}$$

with CC denoting the complex conjugate expression. The vector $S_\alpha(\Gamma)$, with $1 \leqslant \alpha \leqslant 4$, is defined through

$$\begin{aligned}
\mathrm{i}S_1(\Gamma) &= \frac{1}{\sqrt{2}}\,(\mu_3 - \mathrm{i}\mu_1)\,(\alpha_1\cos\theta - \alpha_2\sin\theta) + \mathrm{CC}\\
\mathrm{i}S_2(\Gamma) &= \frac{1}{\sqrt{2}}\,(\mu_3 - \mathrm{i}\mu_1)\,(\alpha_1\sin\theta + \alpha_2\cos\theta) + \mathrm{CC}\\
-\mathrm{i}S_3(\Gamma) &= \frac{1}{\sqrt{2}}\,(\mu_4 - \mathrm{i}\mu_2)\,(\alpha_1\cos\theta - \alpha_2\sin\theta) + \mathrm{CC}\\
-\mathrm{i}S_4(\Gamma) &= \frac{1}{\sqrt{2}}\,(\mu_4 - \mathrm{i}\mu_2)\,(\alpha_1\sin\theta + \alpha_2\cos\theta) + \mathrm{CC}\,.
\end{aligned} \tag{5.9}$$

For the case considered in equation (2.9), throughout equations (5.5)–(5.9), we calculate the mean number of photons and the mean square fluctuations. We used the parameters $\lambda = 1$ and $\gamma = 0.1$ of the model, and generalized Schrödinger even and odd cat states with amplitudes $\alpha_1 = 0.3 + 0.1\mathrm{i}$ and $\alpha_2 = 0.2 + 0.1\mathrm{i}$. The results are displayed in figures 7–9.

Specifically, for the Schrödinger even cat state, in figure 7, the expected values of the number of photons are shown, together with the behaviour of the vacuum-like state of the system. These mean values are oscillatory functions of time, with the worthwhile result that the average value in the number of photons can be smaller, for the selected amplitudes, than those of the vacuum-like state.

For the Schrödinger odd cat state, in figure 8, the mean values of the number of photons are shown, together with the behaviour of the vacuum-like state of the system. These mean values are also oscillatory functions of time, but in these cases the average values in the number of photons for the selected amplitudes are always greater than those of the vacuum-like state.

To determine the type of statistics of the photon distribution function, it is necessary to calculate the ratio $\big(\Delta N_j\big)^2_\pm/\langle N_j\rangle_\pm$; this can be done through equations (5.5)–(5.9). The

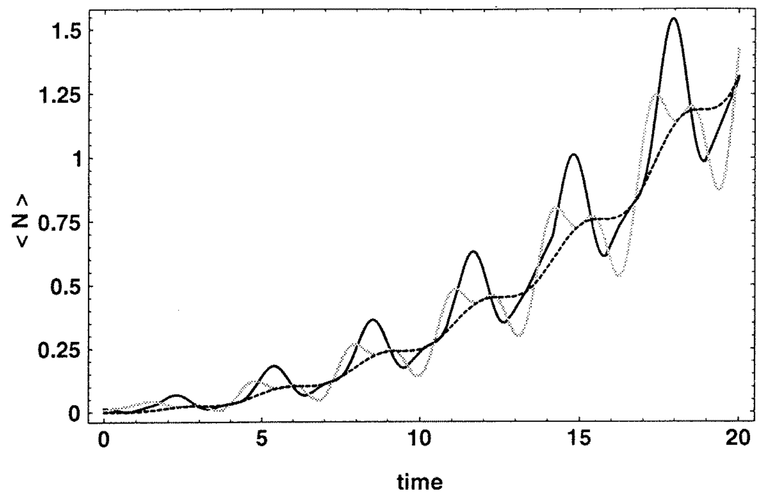

Figure 7. The mean values of the number of photons for the even state, as a function of time. The first mode is plotted in (*a*), and the second in (*b*). The behaviour for the generalized vacuum state is shown with a broken curve. The amplitudes of the state are $\alpha_1 = 0.3 + \mathrm{i}0.1$ and $\alpha_2 = 0.2 + \mathrm{i}0.1$, and the parameters of the model are $\lambda = 1$ and $\gamma = 0.1$.

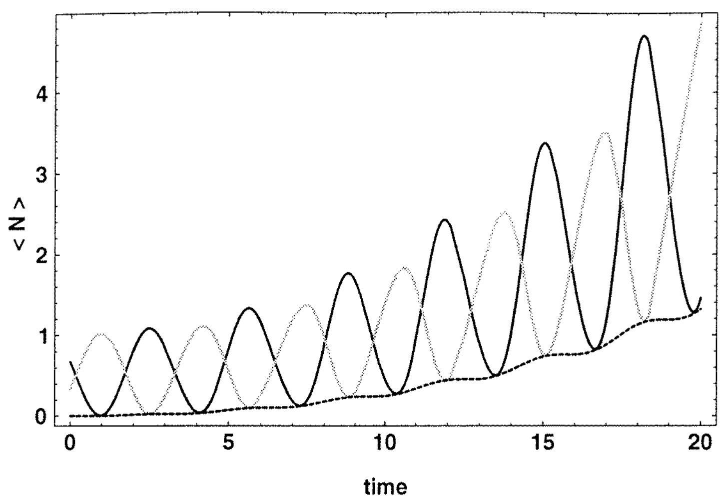

Figure 8. The mean values of the number of photons for the odd state, as a function of time. The first mode is indicated by a full curve and the second mode by a full gray curve. The corresponding plot for the generalized vacuum-like state is shown with a broken curve. The amplitudes and parameters are the same as in figure 7.

results are displayed for the even case in figure 9, and for the odd in figure 10. It is found that the photon distribution function is super-Poissonian for the even case in both modes, while for the odd case there are some intervals of time in which the distribution function oscillates between super-Poissonian and sub-Poissonian statistics.

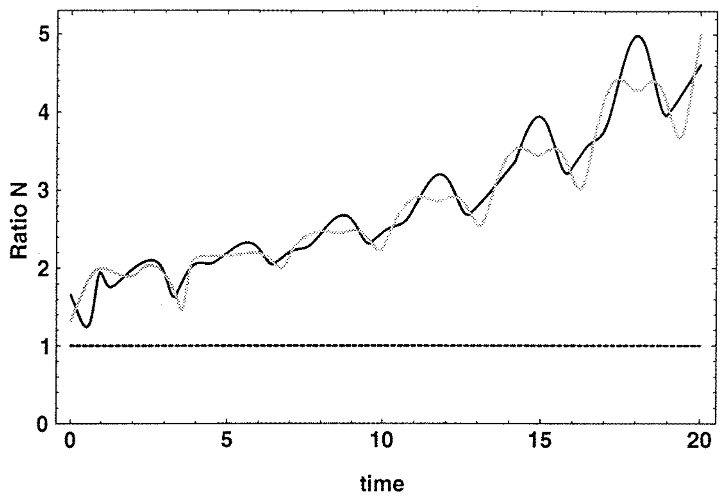

Figure 9. Evolution of the ratio of the mean-squared fluctuation of the number of photons to its mean value for the even state, with the same amplitudes and parameters given in figure 7. The first and second modes are displayed, with full and full grey curves, respectively. The value 1 is indicated with a broken line as a reference.

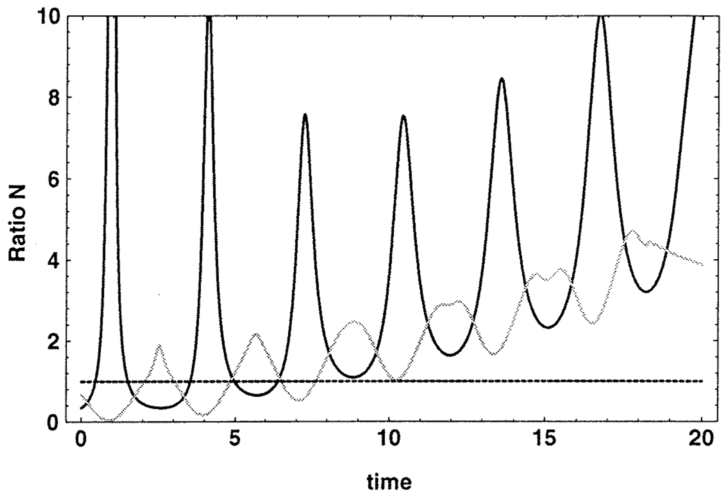

Figure 10. Evolution of the ratio of the mean-squared fluctuation of the number of photons to its mean value for the odd state. Again the first mode is displayed with a full curve, and the second with a full grey curve. The value 1 is shown with a broken line. The amplitudes and parameters are the same as in figure 7.

6. Conclusions

In this paper, for the generalized Schrödinger cat states, we have found analytical results for the quadrature components, equations (3.5) and (3.6), the Husimi (equation (4.6)) and Wigner (eqautions (4.15) and (4.16)), quasidistribution functions and the photon distribution function, (5.3) and (5.4). It is important to emphasize that for the generalized Schrödinger cat states the quadrature dispersions depend on the time-dependent parameter $\lambda_0(t)$, which was not the case for squeezed states of the generalized oscillator [29]. All the analytical results are functions of the symmetry group parameters μ_k and θ. The photon distribution function preserves the behaviour of standard Schrödinger cat states.

The model discussed in this paper demonstrates that two-mode Schrödinger cat states can have squeezed coherent components. Specific time dependence of the oscillator parameters yields squeezing and correlation phenomena for one of the quadrature components of each mode. The collapses and revivals of the mean photon numbers are shown. Statistics of the photon distribution function are found by means of the generating function techniques as explained in the appendix. A sub-Poissonian or super-Poissonian behaviour is found depending on the range of the parameters of the generalized oscillator. A wavy character is exhibited for the distribution functions of the Schrödinger even and odd cat states. For the odd case, these oscillations are enhanced. The model may represent the non-stationary Casimir effect which influences the initial field taken not in the form of coherent or squeezed states, but in the form of even or odd coherent states. The creation of photon and quadrature squeezing in a resonator with moving boundaries at constant velocity [24] is different quantitatively in our model, though qualitatively the results are similar. It is interesting to consider evolution of other non-classical states under the influence of the non-stationary Casimir effect in the context of an exactly solvable model of the generalized oscillator.

Acknowledgment

The authors would like to thank Margarita Man'ko for her kind assistance.

Appendix

The characteristic function for arbitrary $2N$-dimensional Schrödinger cat states is defined by the Fourier transform of the Wigner distribution function of the physical system in question

$$\mathcal{C}_{\pm}(w,t) = \left(\frac{1}{2\pi}\right)^N \int \mathrm{d}^{2N} Q \exp(\mathrm{i}\widetilde{w}Q)\, W_{\pm}(Q,t) \tag{A.1}$$

where in the previous expression we define the $2N$-vectors

$$Q = \begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{pmatrix} \tag{A.2a}$$

$$w = \begin{pmatrix} \boldsymbol{\tau} \\ \boldsymbol{\sigma} \end{pmatrix}. \tag{A.2b}$$

The corresponding Weyl–Wigner distribution function is given in (4.16) and it was rewritten in terms of Q.

To calculate expression (A1), it is convenient to introduce the function

$$\mathcal{C}_{\Gamma}(w,t) = \left(\frac{1}{2\pi}\right)^N \int \mathrm{d}^{2N} Q \exp(\mathrm{i}\widetilde{w}Q)\, W_{\Gamma}(Q,t) \tag{A.3}$$

where we have defined

$$\Gamma = \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta}^* \end{pmatrix}. \tag{A.4}$$

$W_\Gamma(Q,t)$ is the function defined by (4.9) and in the notation of this appendix can be written as

$$W_\Gamma(Q,t) = 2^N \exp\left(\frac{\widetilde{\Gamma}\,\Sigma_x\,\Gamma - \widetilde{\Gamma}^*\Gamma}{2}\right) \exp\left(-\widetilde{Q}\,\widetilde{\Lambda}^*_{pq}\,\Lambda_{pq}\,Q \,+\, 2\widetilde{\Gamma}\,\Lambda^*_{pq}\,Q\right) \tag{A.5}$$

with

$$\Sigma_x \equiv \begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix}.$$

Substituting (A.5) into (A.3), we arrive at an integral of a generalized Gaussian function, which can be calculated through the well known result

$$\int \mathrm{d}^{2N} X \exp(-\widetilde{X} A X + \widetilde{B} X) = \frac{\pi^N}{\sqrt{\det A}} \exp\left(\tfrac{1}{4}\widetilde{B} A^{-1} B\right).$$

After some straightforward algebraic manipulations we get

$$\mathcal{C}_\Gamma(w,t) = \mathcal{C}_\Gamma(0) \exp\left(-\tfrac{1}{4}\,\widetilde{w}\,R(t)\,w + \mathrm{i}\,\widetilde{S}(\Gamma,t)\,w\right), \tag{A.6}$$

where we have defined

$$\mathcal{C}_\Gamma(0) = \exp\left(\frac{\widetilde{\Gamma}\,\Sigma_x\,\Gamma - \widetilde{\Gamma}^*\Gamma}{2}\right) \tag{A.7a}$$

$$R(t) = \{\widetilde{\mathbf{\Lambda}}^*_{pq}(t)\,\mathbf{\Lambda}_{pq}(t)\}^{-1} \tag{A.7b}$$

$$\widetilde{S}(\Gamma,t) = \widetilde{\Gamma}\,\widetilde{\mathbf{\Lambda}}^{-1}_{pq}(t). \tag{A.7c}$$

It is important to emphasize that all the physical information of the quantum system is determined by the $2N \times 2N$ matrix $\mathbf{\Lambda}_{pq}(t) = \boldsymbol{g}_4\mathbf{\Lambda}$. If one considers the parameter $\Gamma = (\boldsymbol{\alpha}, \boldsymbol{\alpha}^*)$, equation (A.5) corresponds to the characteristic function of the generalized correlated states and it can be used to evaluate the expectation values of products of powers of positions and momenta.

Making the substitution of (4.16) into (A.1), and through (A.3), we get the characteristic function of the Schrödinger cat states, i.e.

$$\mathcal{C}_\pm(w,t) = |\mathcal{N}_\pm|\{\mathcal{C}_{\Gamma_1}(w,t) + \mathcal{C}_{-\Gamma_1}(w,t) \pm (\mathcal{C}_{\Gamma_2}(w,t) + \mathcal{C}_{\Gamma_2}(w,t))\} \tag{A.8}$$

with

$$\Gamma_1 = \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\alpha}^* \end{pmatrix} \qquad \text{and} \qquad \Gamma_2 = \begin{pmatrix} \boldsymbol{\alpha} \\ -\boldsymbol{\alpha}^* \end{pmatrix}.$$

For example, the expectation value of the product $Q_k^n Q_l^m$ can be evaluated through the relation

$$\langle Q_k^n Q_l^m \rangle_\pm = (-\mathrm{i})^{n+m} \frac{\partial^{n+m}}{\partial w_l^m\,\partial w_k^n} \{\mathcal{C}_\pm(w,t)\}|_{w=0}. \tag{A.9}$$

To calculate the mean values of the number of photons, $\langle N_k\rangle$ and its mean square fluctuation $(\Delta N_k)^2$, we express the corresponding operators in terms of the positions and momenta:

$$N_k = \tfrac{1}{2}\left\{p_k^2 + q_k^2 - 1\right\} \tag{A10a}$$

$$N_k^2 = \tfrac{1}{4}\left\{p_k^4 + q_k^4 + 1 + p_k^2 q_k^2 + q_k^2 p_k^2 - 2\left(p_k^2 + q_k^2\right)\right\}. \tag{A.10b}$$

Next we find their Weyl–Wigner representation, and afterwards it is necessary to evaluate similar expressions to (A9). We give the results:

$$\langle Q_l^2\rangle_\pm = \tfrac{1}{2}R_{ll} + 2\,|\mathcal{N}_\pm|^2\,\left\{S_l^2(\Gamma_1) \pm \mathcal{C}_{\Gamma_2}(0)S_l^2(\Gamma_2)\right\} \tag{A.11}$$

$$\langle Q_l^4\rangle_\pm = \tfrac{3}{4}R_{ll}^2 + 2\,|\mathcal{N}_\pm|^2\,\left\{3S_l^2(\Gamma_1)R_{ll} + S_l^4(\Gamma_1) \pm \mathcal{C}_{\Gamma_2}(0)\left[3S_l^2(\Gamma_2)R_{ll} + S_l^4(\Gamma_2)\right]\right\} \tag{A.12}$$

$$\begin{aligned}\langle Q_l^2 Q_m^2\rangle_\pm = {}& \tfrac{1}{2}R_{lm}^2 + \tfrac{1}{4}R_{ll}R_{mm} + |\mathcal{N}_\pm|^2\,\left\{4R_{lm}S_l(\Gamma_1)S_m(\Gamma_1) + R_{mm}S_l^2(\Gamma_1)\right.\\ &\left. + R_{ll}S_m^2(\Gamma_1) + 2S_l^2(\Gamma_1)S_m^2(\Gamma_1)\right\} \pm |\mathcal{N}_\pm|^2\,\mathcal{C}_{\Gamma_2}(0)\left\{4R_{lm}S_l(\Gamma_2)S_m(\Gamma_2)\right.\\ &\left. + R_{mm}S_l^2(\Gamma_2) + R_{ll}S_m^2(\Gamma_2) + 2S_l^2(\Gamma_2)S_m^2(\Gamma_2)\right\}\end{aligned} \tag{A.13}$$

where for simplicity we have not written the time dependence for R and S. It is important to emphasize that these expressions are very useful. For example, the dispersion matrices are given by

$$\sigma_{ll}^{\pm} = \left\langle Q_l^2\right\rangle_\pm \tag{A.14}$$

because $\langle Q_l\rangle_\pm = 0$. Therefore through (A.10a), we can immediately conclude that the mean value of N_k is given by the relation (5.5). Next we write the result that we get for the mean squared fluctuation:

$$\begin{aligned}(\Delta N_j)^2 = \langle N_j^2\rangle - \langle N_j\rangle^2 = {}& \tfrac{1}{8}(R_{j,\,j} + R_{N+j,\,N+j})^2 + \tfrac{1}{4}(R_{N+j,\,j}^2 - R_{N+j,\,N+j}R_{j,\,j} - 1)\\ & + |\mathcal{N}_\pm|^2\,\{R_{N+j,\,N+j}S_{N+j}^2(\Gamma_1) + R_{j,\,j}S_j^2(\Gamma_1) + 2R_{N+j,\,j}S_{N+j}(\Gamma_1)S_j(\Gamma_1)\\ & + \tfrac{1}{2}(S_{N+j}^2(\Gamma_1) + S_j^2(\Gamma_1))^2\} \pm |\mathcal{N}_\pm|^2\,\mathcal{C}_{\Gamma_2}(0)\{R_{N+j,\,N+j}S_{N+j}^2(\Gamma_1)\\ & + R_{j,\,j}S_j^2(\Gamma_1) + 2R_{N+j,\,j}S_{N+j}(\Gamma_2)S_j(\Gamma_2) + \tfrac{1}{2}(S_{N+j}^2(\Gamma_2) + S_j^2(\Gamma_2))^2\}\\ & - |\mathcal{N}_\pm|^4\,\{S_{N+j}^2(\Gamma_1) + S_j^2(\Gamma_1) \pm \mathcal{C}_{\Gamma_2}(0)(S_{N+j}^2(\Gamma_2) + S_j^2(\Gamma_1))\}^2.\end{aligned} \tag{A.15}$$

Finally, by recalling

$$\boldsymbol{\sigma}_{jj} = \tfrac{1}{2}R_{jj} \tag{A.16a}$$

$$\langle q_j\rangle = S_{N+j}(\Gamma_1) \tag{A.16b}$$

$$\langle p_j\rangle = S_j(\Gamma_1) \tag{A.16c}$$

we arrive at expression (5.6) for the mean-squared fluctuations.

References

[1] Dodonov V V, Malkin I A and Man'ko V I 1974 *Physica* **72** 597
[2] Schrödinger E 1935 *Naturwissenschaften* **23** 844
[3] Yurke B and Stoler D 1986 *Phys. Rev. Lett.* **57** 13
[4] Perina J 1984 *Quantum Statistics of Linear and Nonlinear Optical Phenomena* (Dordrecht: Reidel)
[5] Gerry C C and Hach III E E 1993 *Phys. Lett.* **174A** 185

[6] Bužek V, Vidiella-Barranco A and Knight P L 1992 *Phys. Rev.* A **45** 6570
[7] Nieto M M and Truax D R 1993 *Phys. Rev. Lett.* **71** 2843
[8] Davidovich L, Maali A, Brune M, Raimond J M and Haroche S 1993 *Phys. Rev. Lett.* **71** 2360
[9] Haroche S 1995 *Nuovo Cimento* B **110** 545
[10] Ansari N A, Di Fiore L, Man'ko M A, Man'ko V I, Solimeno S and Zaccaria F 1994 *Phys. Rev.* A **49** 2151
[11] Stoler D 1971 *Phys. Rev.* D **4** 2309
[12] Hollenhorst J N 1979 *Phys. Rev.* D **19** 1669
[13] Walls D F 1983 *Nature* **306** 141
[14] Barnett S M and Knight P L 1985 *J. Opt. Soc. Am.* B **2** 467
Caves C M and Schumaker B L 1985 *Phys. Rev.* A **31** 3068
Schumaker B L and Caves C M 1985 *Phys. Rev.* A **31** 3093
Caves C M, Chang Zhu, Milburn G L and Schleich W 1991 *Phys. Rev.* A **43** 3854
[15] Bužek V, Moya-Cessa H, Knight P L and Phoenix J J D 1992 *Phys. Rev.* A **45** 8190
[16] Yurke B, McCall S L and Klauder J R 1986 *Phys. Rev.* A **33** 4033
[17] Schrade G, Akulin V M, Man'ko V I and Schleich W 1993 *Phys. Rev.* A **48** 2398
[18] Ansari N A and Man'ko V I 1994 *Phys. Rev.* A **50** 1942
[19] Dodonov V V, Man'ko V I and Nikonov D E 1995 *Phys. Rev.* A **51** 3328
[20] Castaños O and López-Peña R 1992 *J. Phys. A: Math. Gen.* **25** 6685
[21] Castaños O, López-Peña R and Man'ko V I 1994 *J. Phys. A: Math. Gen.* **27** 1751
[22] Man'ko V I 1992 *Symmetries in Physics, Proc. Int. Symp. in Honor of Professor Moshinsky (Cocoyoc, Morelos 1991)* ed A Frank and K B Wolf (Berlin: Springer) p 311
[23] Dodonov V V, Klimov A B and Man'ko V I 1990 *Phys. Lett.* **149A** 225
Dodonov V V, Klimov A B and Nikonov D E 1993 *Phys. Rev.* A **47** 4422
[24] Jáuregui R, Villarreal C and Hacyan S 1995 *Mod. Phys. Lett.* A **10** 619
Villarreal C, Hacyan S and Jáuregui R 1995 *Phys. Rev.* A **52** 601
[25] Law C K 1994 *Phys. Rev.* A **49** 433
[26] Man'ko V I 1994 *Proc. Third Int. Workshop on Squeezed States and Uncertainty Relations (Baltimore 1993) (NASA Conf. Publication* **3270***)* ed D Han, M Rubin and W W Zachary p 365
[27] Glauber R J 1963 *Phys. Rev. Lett.* **10** 84
[28] Dodonov V V and Man'ko V I 1989 *Invariants and Evolution of Nonstationary Quantum Systems (Proc. Lebedev Physical Institute* **183***)* ed M A Markov (Commack, NY: Nova Science)
[29] Castaños O, López-Peña R and Man'ko V I 1994 *Phys. Rev.* A **50** 5209